

# Nil Clean Involutions

Janez Šter

Faculty of Mechanical Engineering  
University of Ljubljana

janez.ster@fs.uni-lj.si

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## Abstract

We prove that if an involution in a ring is the sum of an idempotent and a nilpotent then the idempotent in this decomposition must be 1. As a consequence, we completely characterize weakly nil-clean rings introduced recently in [Breaz, Danchev and Zhou, Rings in which every element is either a sum or a difference of a nilpotent and an idempotent, J. Algebra Appl., DOI: 10.1142/S0219498816501486].

In this note rings are unital.  $U(R)$ ,  $\text{Id}(R)$ ,  $\text{Nil}(R)$  and  $\text{Nil}^*(R)$  stand for the set of units, the set of idempotents, the set of nilpotents and the upper nilradical of a ring  $R$ , respectively.  $\mathbb{Z}_n$  stands for the set of integers modulo  $n$ . An *involution* in a ring means an element  $a$  satisfying  $a^2 = 1$ .

Following [2], we say that an element in a ring is *nil clean* if it is the sum of an idempotent and a nilpotent, and a ring is nil clean if every element is nil clean. The main result in this note is the following:

**Proposition 1.** *Let  $R$  be a ring with an involution  $a \in R$ . If  $a$  is the sum of an idempotent  $e$  and a nilpotent  $q$  then  $e = 1$ . In particular, every nil clean involution in a ring is unipotent (i.e. 1 plus a nilpotent).*

*Proof.* Write  $a = e + q$  with  $e \in \text{Id}(R)$  and  $q \in \text{Nil}(R)$ , and denote  $f = 1 - e \in \text{Id}(R)$  and  $r = q(1 + q) \in \text{Nil}(R)$ . From  $fq = f(a - e) = fa$  we compute  $fr = fq(1 + q) = fa(1 + a - e) = fa(f + a) = faf + fa^2 = faf + f$ , and similarly  $rf = faf + f$ . Hence  $fr = rf$ , so that  $r$  is a nilpotent which commutes with  $f$ ,  $e$ ,  $q$  and  $a$ . Accordingly,

$$f = fa^2 = fqa = fr(1 + q)^{-1}a = f(1 + q)^{-1}a \cdot r$$

is a nilpotent and hence  $f = 0$ , as desired.  $\square$

Following [1], we say that a ring is *weakly nil-clean* if every element is either a sum or a difference of a nilpotent and an idempotent.

**Lemma 2.** *If  $R$  is a weakly nil-clean ring with  $2 \in U(R)$  then  $R/\text{Nil}^*(R) \cong \mathbb{Z}_3$ .*

*Proof.* Choose any idempotent  $e \in \text{Id}(R)$ , and set  $a = 1 - 2e$ . By assumption, either  $a$  or  $-a$  is nil clean. If  $a$  is nil clean then, since  $a^2 = 1$ , Proposition 1 gives that  $a - 1 = -2e$  is a nilpotent, so that  $e$  is a nilpotent and hence  $e = 0$ . Similarly, if  $-a$  is nil clean then, since  $(-a)^2 = 1$ , Proposition 1 gives that  $-a - 1 = -2(1 - e)$  is a nilpotent, so that  $1 - e$  is a nilpotent and hence  $e = 1$ . This proves that  $R$  has only trivial idempotents. Accordingly, since  $R$  is weakly nil clean, every element of  $R$  must be either  $q$  or  $1 + q$  or  $-1 + q$  for some  $q \in \text{Nil}(R)$ . From this, one quickly obtains that  $\text{Nil}(R)$  must actually form an ideal in  $R$ , so that  $R/\text{Nil}^*(R)$  can have only 3 elements and hence  $R/\text{Nil}^*(R) \cong \mathbb{Z}_3$ , as desired. (Alternatively, considering that  $R$  is abelian,  $R/\text{Nil}^*(R) \cong \mathbb{Z}_3$  can be also obtained from [1, Theorem 12].)  $\square$

Using the above lemma, we have:

**Theorem 3.** *A ring is weakly nil clean if and only if it is either nil clean or isomorphic to  $R_1 \times R_2$  where  $R_1$  is nil clean and  $R_2/\text{Nil}^*(R_2) \cong \mathbb{Z}_3$ .*

*Proof.* Follows from Lemma 2 together with [1, Theorem 5]. □

**Remark 4.** Proposition 1 can be generalized to arbitrary algebraic elements of order 2 as follows. Let  $R$  be an algebra over a commutative ring  $k$ , and let  $a \in R$  be an element satisfying  $\alpha a^2 + \beta a + \gamma = 0$ , with  $\alpha, \beta, \gamma \in k$ , and suppose that  $a = e + q$  with  $e \in \text{Id}(R)$  and  $q^n = 0$ . Then one can show that  $r = q(\alpha q + \alpha + \beta)$  is a nilpotent commuting with  $e$ , which yields, similarly as in Proposition 1, that

$$(\alpha + \beta)^n e + (\alpha + \beta)^{n-1} \gamma$$

is also a nilpotent. Note that this result indeed generalizes Proposition 1 (taking  $\alpha = 1$ ,  $\beta = 0$  and  $\gamma = -1$  yields that  $e - 1$  is a nilpotent, so that  $e = 1$ ). However, for orders of algebraicity higher than 2 this argument no longer seems to work.

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## References

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